## **Kinetic Energy**

NAME:

We have stated in class that the kinetic energy of a rigid object that is both moving linearly and rotating about its center of mass is simply the sum of the translational and rotation kinetic energies. Let's prove it.

Imagine an arbitrary object is moving to the right and its center of mass (labeled CM in the diagram) has a velocity of  $v_{cm}$ . At the same time, it is also rotating about its center mass, let's say clockwise with an angular velocity of w.

Figuring out the kinetic energy of the object seems straightforward enough: we just break the object into lots of little pieces (one shown in the diagram as m) and add up all the kinetic energies of the little pieces. The annoying part is that all the little pieces have their own individual velocity, v. However, each little velocity can be broken down into two parts: it is moving to the right with velocity  $v_{cm}$  because the center of mass is moving and it has a tangential velocity of  $v_t$  due to the rotation. So we can say

$$v = v_{cm} + v_t$$

Therefor, the kinetic energy is

$$K = \sum \frac{1}{2}mv^{2} = \sum \frac{1}{2}m(\overrightarrow{v_{cm}} + \overrightarrow{v_{t}})^{2} = \sum \frac{1}{2}m(\overrightarrow{v_{cm}}^{2} + \overrightarrow{v_{t}}^{2} + 2\overrightarrow{v_{cm}} \cdot \overrightarrow{v_{t}})$$

Now it is easier to separate that into three separate summations. The first term is simple, and is just the kinetic energy the object has because its center of mass is moving:

$$\sum \frac{1}{2} m \overrightarrow{v_{cm}^{2}} = \frac{1}{2} \left( \sum m \right) v_{cm}^{2} = \frac{1}{2} M v_{cm}^{2} = K_{translation}$$

For the second term, we have to remember that the tangential velocity for a little mass depends on how far away from the axis it is,  $v_t$ =rw. We also have to remember that there is only one rotation rate because it is a rigid body. This term is just the rotational kinetic energy of the object:

$$\sum \frac{1}{2} m v_t^2 = \frac{1}{2} \sum m(r\omega)^2 = \frac{1}{2} \left( \sum m r^2 \right) \omega^2 = \frac{1}{2} I \omega^2 = K_{rotation}$$

The last term ends up being zero. We have to remember that when we are multiplying two vectors we are taking the dot product  $(v_{cm} \bullet v_t)$ . For every single little mass that makes up the object,  $v_{cm}$  is a constant in both magnitude and direction - it always points horizontally. The tangential velocity depends on the little mass. In general, the dot product of two vectors is the projection of one vector onto the second. So in this case, the dot product is the velocity of the center of mass times the horizontal component of the tangential velocity.

$$\sum_{t=1}^{t} m 2 \overrightarrow{v_{cm}} \cdot \overrightarrow{v_{t}} = \sum_{t=1}^{t} m v_{cm} v_{t} \sin \theta = \sum_{t=1}^{t} m v_{cm} r \omega \sin \theta$$

Notice how the rsinø term is actually the y-coordinate of the little mass, as measured from the center of mass. So we end up with

$$\sum mv_{cm}r\omega\sin\theta = v_{cm}\omega\sum mr\sin\theta = v_{cm}\omega\sum my$$

By the definition of center of mass, that last summation is zero! So we can say that the kinetic energy of an object moving and rotating is simply

$$K = K_{translation} + K_{rotation}$$

